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Ideal Proportional Navigation

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Proportional navigation has been proved to be a useful guidance technique in several surface-to-air and air-to-air homing systems for interception of airborne targets. Besides the familiar pure, true, and generalized proportional navigation guidance laws, a new guidance scheme, called ideal proportional navigation, with commanded acceleration applied in the direction normal to the relative velocity between interceptor and target, is presented. In this study the closed-form solutions of ideal proportional navigation are completely derived for maneuvering and nonmaneuvering targets, and some important characteristics related to the system performance are introduced. Under this scheme the capture criterion is related to the effective proportional navigation constant only, no matter where the initial condition and target maneuver are. With some more energy consumption, this new guidance scheme has a larger capture area and is much more effective than the other schemes.

Nomenclature

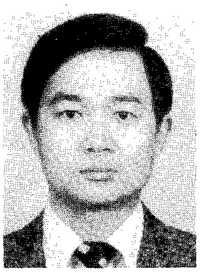
a = acceleration
c = $a_T/v_0\dot{\theta}_0$
r = distance between interceptor and target
T = total time of flight
t = time of flight
V = velocity
v = relative velocity between interceptor and target
Γ = gamma function
θ = angle between line of sight and inertial reference line
λ = effective proportional navigation constant
ϕ = angle between relative velocity and line of sight

Subscripts
c = commanded value
f = final value
M = missile, interceptor
r = parallel to line of sight
T = target
z = normal to the intercept plane
θ = normal to line of sight
0 = initial value
1 = state at maximum r

Superscript
$\dot{}$ = time derivative

Introduction

In traditional proportional navigation, pure proportional navigation (PPN) and true proportional navigation (TPN) have been widely introduced and thoroughly developed for homing guidance. In PPN the commanded acceleration is applied in the direction normal to the interceptor velocity; in TPN the commanded acceleration is applied in the direction normal to the line of sight (LOS) between the interceptor and its target. A qualitative study of PPN was performed by Guelman[1,2] and Becker,[3] and the corresponding closed-form solutions were derived. An early study in TPN was performed by Murtaugh[4] under the assumption of a constant closing rate and a nonmaneuvering target.[4] Then a closed-form solution of TPN was obtained by Guelman,[5] who did not use the varied closing rate as feedback to the commanded acceleration. Finally, a complete analytic solution of TPN with varying closing speed feedback to the commanded acceleration was derived by Yuan and Chern[6]; this solution seems more significant than previous ones. Recently, a generalized proportional



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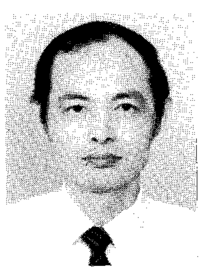

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navigation (GPN) was developed by Yang et al.,[7,8] who applied the commanded acceleration in an arbitrary direction other than the certain direction normal to the LOS.

In this paper a new guidance scheme, ideal proportional navigation (IPN), is presented. In this scheme the commanded acceleration is applied in the direction normal to the relative velocity between the interceptor and its target, and its magnitude is proportional to the product of LOS rate and relative velocity, trying to turn the relative velocity in most effort to the direction of LOS with utmost effort. First, the system configuration is formulated in a polar coordinate system with a nonmaneuvering target for exoatmospheric flight. A specific target maneuver is then considered and is applied in the direction opposite to the commanded acceleration for effective escape during the intercept period. The closed-form solutions are completely derived under this guidance scheme and some important characteristics related to the system performance are obtained from the results. Also, the capture criterion and cumulative velocity increment required are investigated in detail for maneuvering and nonmaneuvering targets. The cumulative velocity increment is an important parameter in exoatmospheric flight and is related to the corresponding propellant mass required for effective intercept.

Solution for a Nonmaneuvering Target

Consider an interceptor of speed V_M pursuing a nonmaneuvering target with speed V_T in the same plane under the guidance law of IPN. We define the relative velocity $\boldsymbol{v}$ between interceptor and its target expressed in polar coordinates as

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{V}_M - \boldsymbol{V}_T = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta \\ &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \end{aligned} \tag{1}$$

In IPN the commanded acceleration $\boldsymbol{a}_c$ is applied in the direction normal to the relative velocity $\boldsymbol{v}$ as depicted in Fig. 1, and its magnitude is proportional to the product of LOS rate and relative velocity, i.e.,

$$\begin{aligned} \boldsymbol{a}_c &= \lambda\dot{\theta}\boldsymbol{e}_z \times \boldsymbol{v} \\ &= \lambda(-r\dot{\theta}^2\boldsymbol{e}_r + \dot{r}\dot{\theta}\boldsymbol{e}_\theta) \end{aligned} \tag{2}$$

where $\boldsymbol{e}_z = \boldsymbol{e}_r \times \boldsymbol{e}_\theta$. Then the equations of relative motion between the interceptor and its target under the guidance law of IPN with a nonmaneuvering target can be described as

$$\ddot{r} - r\dot{\theta}^2 = -\lambda r\dot{\theta}^2 \tag{3a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \lambda\dot{r}\dot{\theta} \tag{3b}$$

As presented in Ref. 6, $\dot{\theta}$ can be solved from Eq. (3b), and the result is

$$\dot{\theta} = \dot{\theta}_0(r/r_0)^{\lambda-2} \tag{4}$$

During the intercept period, $\dot{\theta}$ will approach infinity when $\lambda<2$ and approach zero when $\lambda>2$. The result is the same as that of TPN obtained by Yuan and Chern,[6] as shown in Fig. 2. By substituting Eq. (4) into Eq. (3a), the solution of $\dot{r}$ can be obtained as

$$\begin{aligned} \dot{r}^2 &= -r_0^2\dot{\theta}_0^2(r/r_0)^{2(\lambda-1)} + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2 \\ &= -r_0^2\dot{\theta}_0^2(r/r_0)^{2(\lambda-1)} + v_0^2 \end{aligned} \tag{5}$$

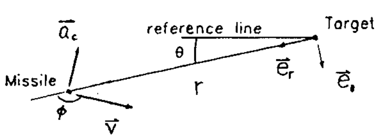

Fig. 1 Planar pursuit geometry.

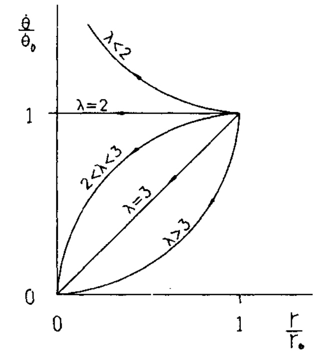

Fig. 2 $\dot{\theta}$ vs r for different values of λ.

where v_0 is the initial magnitude of relative velocity. It is seen from Eq. (5) that the capture criterion for IPN is simply $\lambda>1$. In other words, no matter what the initial condition of $\dot{r}$ is ($\dot{r}_0<0$ or $\dot{r}_0\geq 0$), the intercept can always be achieved successfully when $\lambda>1$. From Eqs. (4) and (5), we find that the magnitude of relative velocity is maintained at a constant, and its direction is turned to the target during the intercept period, i.e.,

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = v_0 \tag{6a}$$

$$r\dot{\theta} = r_0\dot{\theta}_0(r/r_0)^{\lambda-1} = v\sin\phi \tag{6b}$$

$$\dot{r} = v\cos\phi \tag{6c}$$

where the angle ϕ between LOS and relative velocity can be expressed as a function of range-to-go r:

$$\sin\phi/\sin\phi_0 = (r/r_0)^{\lambda-1}$$

The corresponding commanded acceleration can be described as

$$a_c = \lambda v\dot{\theta} = \lambda v_0\dot{\theta}_0(r/r_0)^{\lambda-2} \tag{7}$$

Since the actual responsed acceleration is finite, λ is normally chosen to be >2 to avoid a_c going to infinity.

The t can be solved as a function of r. When $\dot{r}_0\leq 0$, it is

$$t = \int_{r_0}^{r}\frac{\mathrm{d}r}{\dot{r}} = -\int_{r_0}^{r}\frac{\mathrm{d}r}{\sqrt{v_0^2 - r_0^2\dot{\theta}_0^2(r/r_0)^{2\lambda-2}}}$$

Thus, T is

$$\frac{T}{r_0/v_0} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-\sin^2\phi_0 x^{2\lambda-2}}}$$

A general solution of T can be derived as (see Appendix A)

$$T = \frac{(\sin\phi_0)^{(\lambda-2)/(\lambda-1)}}{(\lambda-1)\dot{\theta}_0}\int_{\phi_0}^{\pm\pi}(\sin\phi)^{(2-\lambda)/(\lambda-1)}\,\mathrm{d}\phi$$

$$(+,\ \text{if}\ \phi_0>0; \qquad -,\ \text{if}\ \phi_0<0) \tag{8}$$

It can be simplified to

$$T = (\pm\pi - \phi_0)/\dot{\theta}_0$$

when λ is equal to 2.

The relation between θ and r can be derived as follows. By differentiating Eq. (6b) and combining with Eq. (6c), we have

$$\dot{\phi} = (\lambda-1)\dot{\theta}_0(r/r_0)^{\lambda-2} = (\lambda-1)\dot{\theta} \tag{9}$$

The solution of this differential equation is

$$\theta = \frac{1}{\lambda - 1}(\phi - \phi_0)$$

$$= \frac{1}{\lambda - 1}\left\{\sin^{-1}\left[\sin\phi_0 (r/r_0)^{\lambda - 1}\right] - \phi_0\right\} \tag{10}$$

and its inverse function is

$$\frac{r}{r_0} = \left(\frac{\sin\phi}{\sin\phi_0}\right)^{1/(\lambda-1)} = \left\{\frac{\sin\left[(\lambda-1)\theta + \phi_0\right]}{\sin\phi_0}\right\}^{1/(\lambda-1)}$$

Figure 3 shows a typical relation between r and θ for different values of λ. The final value of θ is

$$\theta_f = \frac{1}{\lambda - 1}(\pm\pi - \phi_0)$$

$$(+,\ \text{if}\ \ \phi_0 > 0; \qquad -,\ \text{if}\ \ \phi_0 < 0) \tag{11}$$

In Appendix A another method is introduced to derive these functions directly by solving the rearranged equations of motion, and some related characteristics are investigated.

Finally, the total cumulative velocity increment required until intercept can be expressed as

$$\Delta V = \int_0^T |a_c|\ \mathrm{d}t$$

$$= \left[\lambda/(\lambda - 1)\right] v(\pi - |\phi_0|) \tag{12}$$

We find that the ΔV required for IPN is greater than that required for TPN, which is equal to $[\lambda/(\lambda-1)]\, v\, \sin(\pi - |\phi_0|)$.[9]

Solution for a Maneuvering Target

For the case of a maneuvering target we assume that the target maneuver is always normal to the relative velocity, trying to resist the relative velocity turning to the direction of LOS. Then we have the equations of relative motion between the interceptor and its target:

$$\dot{v} = a_c - a_T = \lambda\dot{\theta} e_z \times v - a_T e_z \times e_v$$

or

$$\ddot{r} - r\dot{\theta}^2 = -\lambda r\dot{\theta}^2 + \frac{a_T}{v} r\dot{\theta} \tag{13a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \lambda\dot{r}\dot{\theta} - \frac{a_T}{v}\dot{r} \tag{13b}$$

Because both the missile commanded acceleration and the target maneuver are applied in the direction normal to the relative velocity, the magnitude of relative velocity v is obviously

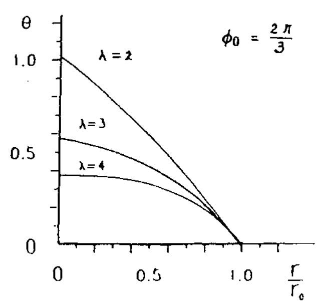

Fig. 3 θ vs r for different values of λ.

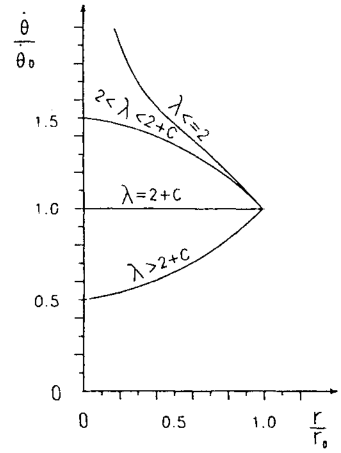

Fig. 4 $\dot{\theta}$ vs r for different values of λ with a maneuvering target.

maintained at constant during the intercept period (see Appendix B). From Eq. (13b) we can solve for $\dot{\theta}$ first (see Ref. 6):

$$\dot{\theta} = \dot{\theta}_0\left\{\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{c}{\lambda - 2}\left[1 - \left(\frac{r}{r_0}\right)^{\lambda - 2}\right]\right\} \quad \text{if} \quad \lambda \neq 2 \tag{14a}$$

$$= \dot{\theta}_0\left(1 - c\,\ln\frac{r}{r_0}\right) \qquad \text{if} \quad \lambda = 2 \tag{14b}$$

in which a positive value of $c(a_T/v_0\dot{\theta}_0)$ is considered, to try to prevent the relative velocity turn to the direction of LOS. It can be seen from Eqs. (14) that the response can be separated into two parts: One part is induced by the initial LOS rate, and the other part is induced by the maneuver of target. The value of $\dot{\theta}$ will aproach infinity during the intercept period when $\lambda \leq 2$ and approaches $a_T/[(\lambda-2)v]$ when $\lambda > 2$, as shown in Fig. 4. Thus, the LOS rate will never approach zero due to target maneuver.

Next we try to solve $\dot{r}$ as follows. Substituting Eqs. (14) into Eq. (13a), the solution of $\dot{r}$ can be obtained as

$$\dot{r}^2 = r_0^2\dot{\theta}_0^2\left\{-\left(1 - \frac{c}{\lambda-2}\right)^2\left(\frac{r}{r_0}\right)^{2\lambda-2} - \frac{2c}{\lambda-2}\left(1 - \frac{c}{\lambda-2}\right)\left(\frac{r}{r_0}\right)^{\lambda} - \frac{c^2}{(\lambda-2)^2}\left(\frac{r}{r_0}\right)^2\right\} + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2 \quad \text{if} \quad \lambda \neq 2 \tag{15a}$$

$$= r_0^2\dot{\theta}_0^2\left\{-c^2\left(\frac{r}{r_0}\right)^2\left(\ln\frac{r}{r_0}\right)^2 + 2c\left(\frac{r}{r_0}\right)^2\ln\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^2\right\} + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2 \quad \text{if} \quad \lambda = 2 \tag{15b}$$

Again the magnitude of $v(\sqrt{\dot{r}^2 + r^2\dot{\theta}^2})$ can be proven to be a constant (v_0) during the intercept period from Eqs. (14) and (15). In this case we find that the capture criterion is also

$$\lambda > 1 \tag{16}$$

In other words, even with a maneuvering target the intercept can still be achieved successfully, no matter what the initial condition and target maneuver are (even $\dot{r}_0 > 0$). The acceleration command in this case can be expressed as

$$a_c = \lambda v\dot{\theta} = \lambda v_0\dot{\theta}_0\left\{\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{c}{\lambda-2}\left[1 - \left(\frac{r}{r_0}\right)^{\lambda-2}\right]\right\} \quad \text{if} \quad \lambda \neq 2 \tag{17a}$$

$$= 2v_0\dot{\theta}_0\left(1 - c\,\ln\frac{r}{r_0}\right) \qquad \text{if} \quad \lambda = 2 \tag{17b}$$

The commanded acceleration approaches infinity with $\lambda \le 2$ and approaches $[\lambda/(\lambda-2)]a_T$ with $\lambda > 2$. Therefore, in application λ is normally chosen to be >2 to avoid a_c (or $\dot\theta$) from going to infinity. Furthermore, if λ is selected as greater than $2+c$, then the magnitude of a_c (or $\dot\theta$) is monotonously decreased during the intercept period.

The relation between ϕ and r can be derived as follows. From the definition of the angle ϕ, $\sin\phi = r\dot\theta/v$, and using the results in Eqs. (14), we have

$$\frac{\sin\phi}{\sin\phi_0} = \left(\frac{r}{r_0}\right)\left\{\frac{c}{\lambda-2} + \left[1-\frac{c}{\lambda-2}\right]\left(\frac{r}{r_0}\right)^{\lambda-2}\right\} \quad \text{if} \quad \lambda \ne 2 \tag{18a}$$

$$= \left(\frac{r}{r_0}\right)\left[1 - c\,\ln\frac{r}{r_0}\right] \quad \text{if} \quad \lambda = 2 \tag{18b}$$

Therefore, the relative velocity v will always turn to head to the target during the intercept period (i.e., $|\phi|$ will approach π). By differentiating Eqs. (18) with respect to time, we have

$$\dot\phi = \dot\theta_0\left\{(\lambda-1)\left[\frac{c}{\lambda-2} + \left(1-\frac{c}{\lambda-2}\right)\left(\frac{r}{r_0}\right)^{\lambda-2}\right] - c\right\} \quad \text{if} \quad \lambda \ne 2 \tag{19a}$$

$$= \dot\theta_0\left\{\left(1 - c\,\ln\frac{r}{r_0}\right) - c\right\} \quad \text{if} \quad \lambda = 2 \tag{19b}$$

The value of $\dot\phi$ approaches infinity at intercept with $\lambda \le 2$ and approaches

$$[1/(\lambda-2)]\big/(a_T/v)$$

with $\lambda > 2$. From Eqs. (19) we also have

$$\theta = \frac{1}{\lambda-1}\left\{\phi - \phi_0 + \frac{a_T}{v}\,t\right\} \tag{20}$$

where t is the time of flight that can be obtained as a function of r (see Appendix B). Figure 5 shows a typical variation of θ with respect to r for different values of c. The final LOS angle θ_f is

$$\theta_f = \frac{\pm\pi - \phi_0 + (a_T/v)T}{\lambda-1}$$

$$(+, \ \text{if} \ \phi_0 > 0; \quad -, \ \text{if} \ \phi_0 < 0)$$

where T is the total time of flight and is derived in Appendix B. Finally, the total cumulative velocity increment ΔV required until intercept can be obtained as

$$\Delta V = \int_0^T |a_c|\,\mathrm{d}t$$

$$= \frac{\lambda}{\lambda-1}\left\{v(\pi - |\phi_0|) + T|a_T|\right\} \tag{21}$$

In this case the cumulative velocity increment required is increased with an additional term due to target maneuver.

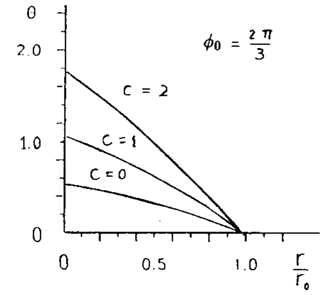

Fig. 5 θ vs r for different values of c with $\lambda = 3$.

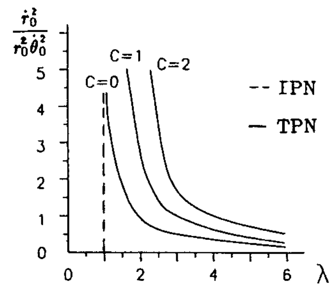

Fig. 6 Capture area for IPN and TPN.

Discussion

In IPN the magnitude of the relative velocity is maintained at a constant and its direction is turned to head to the target during the intercept period. The capture criterion for IPN is simply $\lambda > 1$ to achieve successful intercept of the target, no matter what the initial condition and target maneuver are. While in TPN, the initial condition and target maneuver will affect the capture capability. It is shown that IPN has a larger capture area than TPN, as depicted in Fig. 6. Successful intercept of the target is restricted to the right-hand-side region of each curve representing the capture boundary of different schemes. Because of the limitation of the interceptor acceleration in actual application, λ is normally chosen to be >2 to avoid a_c (or $\dot\theta$) approaching infinity. Furthermore, if λ is selected to be greater than $2+c$, then the commanded acceleration a_c will be monotonously decreased when r is decreased during the intercept period. The target maneuver will not affect the capture capability, but will increase the deflected angle of LOS and the cost of cumulative velocity increment required.

The response of the LOS rate for IPN is similar to that for TPN, but the response of the range-to-go rate for IPN is quite different from that for TPN due to the additional commanded acceleration component in the direction of LOS. The closing rate is monotonously increasing in IPN when the range-to-go is decreasing during the intercept period, whereas it is monotonously decreasing in TPN. Thus, IPN intercepts the target faster than TPN under the same initial conditions, and the cumulative velocity increment required for IPN is obviously greater than that required for TPN.

Furthermore, IPN is more flexible and practical than GPN. In GPN the angle between the commanded acceleration and the normal direction of LOS is a fixed nonzero value during the intercept period. In IPN the commanded acceleration is always normal to the relative velocity; thus, the corresponding angle between the commanded acceleration and the normal direction of LOS is time varied and related to the angle ϕ during the intercept period. Compared to the result of GPN, we find that IPN has a larger capture area and can be considered as a near-optimal solution of GPN.

Conclusion

A new guidance concept, ideal proportional navigation, is introduced in this paper. The closed-form solutions are completely derived, and some important characteristics are investigated. Under this scheme the capture criterion is related to the effective proportional navigation constant only, no matter what the initial condition and target maneuver are. With some more energy consumption, it has a larger capture area and is much more effective than the other schemes.

Appendix A: Intercept of a Nonmaneuvering Target

From Eqs. (1) and (2), we have

$$\dot{v} = \dot{v}e_v + v\dot{e}_v = \lambda\dot{\theta}ve_z \times e_v \tag{A1}$$

Then we obtain

$$\dot{v} = 0$$
$$\dot{\phi} = (\lambda - 1)\dot{\theta} \tag{A2}$$

where ϕ is the angle between the LOS and the relative velocity, as shown in Fig. 1. From Eq. (A2) we find that the magnitude of relative velocity between the interceptor and its target is unchanged during the intercept period and the relation between ϕ and θ can be derived as

$$\phi = (\lambda - 1)\theta + \phi_0 \tag{A3}$$

where ϕ_0 is the initial condition of angle ϕ. Also, from Eqs. (6) and (A2) we have

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = (\lambda - 1)\frac{\tan\phi}{r}$$

The solution of this differential equation is

$$\frac{\sin\phi}{\sin\phi_0} = \left(\frac{r}{r_0}\right)^{\lambda-1} \tag{A4}$$

From Eq. (A4) we can also obtain the same result as that derived in Eqs. (4) and (5). Substituting Eq. (4) into Eq. (A2), we obtain

$$\dot{\phi} = (\lambda - 1)\dot{\theta} = (\lambda - 1)\dot{\theta}_0\left(\frac{\sin\phi}{\sin\phi_0}\right)^{(\lambda-2)/(\lambda-1)}$$

The solution of this differential equation is

$$t = \frac{(\sin\phi_0)^{(\lambda-2)/(\lambda-1)}}{(\lambda-1)\dot{\theta}_0}\int_{\phi_0}^{\phi}(\sin\phi)^{(2-\lambda)/\lambda-1)}\,d\phi$$

and the total time of flight until intercept in Eq. (8) can be derived. For the special case of $|\phi_0| = \pi/2$ (i.e., $\dot{r}_0 = 0$), the total time of flight can be simply expressed as

$$T = \frac{1}{(\lambda-1)|\dot{\theta}_0|}\int_{\pi/2}^{\pi}(\sin\phi)^{(2-\lambda)/(\lambda-1)}\,d\phi$$
$$= \frac{\sqrt{\pi}}{2(\lambda-1)|\dot{\theta}_0|}\frac{\Gamma[1/2(\lambda-1)]}{\Gamma[\lambda/2(\lambda-1)]} \tag{A5}$$

Appendix B: Intercept of a Maneuvering Target

From the equation of relative motion with a maneuvering target,

$$\dot{v} = (\lambda v\dot{\theta} - a_T)e_z \times e_v \tag{B1}$$

we have

$$\dot{v} = 0$$
$$\dot{\phi} = (\lambda - 1)\dot{\theta} - \frac{a_T}{v} \tag{B2}$$

Thus, the magnitude of relative velocity between the interceptor and its target is always maintained at a constant during the intercept period and the relation between ϕ and θ is

$$\theta = \frac{1}{\lambda - 1}\left\{\phi - \phi_0 + \frac{a_T}{v}t\right\} \tag{B3}$$

where the total time of flight can be derived as follows:

When $\dot{r}_0 \le 0$: By definition, we have

$$t = \int_{r_0}^{r}\frac{dr}{\dot{r}} = -\frac{r_0}{v_0}\int_1^{r/r_0}\frac{d(r/r_0)}{\sqrt{1-\sin^2\phi_0(r/r_0)^2(\dot{\theta}/\dot{\theta}_0)^2}} \tag{B4}$$

Then the total time of flight until intercept can be obtained as

$$\frac{T}{r_0/v_0} = \int_0^1\frac{dx}{\sqrt{1-\sin^2\phi_0 x^2[c/\lambda-2+(1-c/\lambda-2)x^{\lambda-2}]^2}} \quad \text{if} \quad \lambda \neq 2$$
$$= \int_0^1\frac{dx}{\sqrt{1-\sin^2\phi_0 x^2[1-c\,\ln x]^2}} \quad \text{if} \quad \lambda = 2 \tag{B5}$$

When $\dot{r}_0 > 0$: First, the maximum range r_1, occurring when $\dot{r}$ goes to zero, can be obtained as

$$\frac{r_1}{r_0}\left[\frac{c}{\lambda-2} + \left(1 - \frac{c}{\lambda-2}\right)\left(\frac{r_1}{r_0}\right)^{\lambda-2}\right] = |\csc\phi_0| \quad \text{if} \quad \lambda \neq 2$$
$$\frac{r_1}{r_0}\left[1 - c\,\ln\frac{r_1}{r_0}\right] = |\csc\phi_0| \quad \text{if} \quad \lambda = 2$$

Then the total time of flight until intercept can be expressed as

$$T = \frac{r_0}{v_0}\int_1^{r_1/r_0}\frac{dx}{\sqrt{1-\sin^2\phi_0 x^2[c/\lambda-2+(1-c/\lambda-2)x^{\lambda-2}]^2}}$$
$$+ \frac{r_1}{v_0}\int_0^1\frac{dx}{\sqrt{1-x^2[c/\lambda-2+(1-c/\lambda-2)x^{\lambda-2}]^2}} \quad \text{if} \quad \lambda \neq 2$$
$$= \frac{r_0}{v_0}\int_1^{r_1/r_0}\frac{dx}{\sqrt{1-\sin^2\phi_0 x^2[1-c\,\ln x]^2}}$$
$$+ \frac{r_1}{v_0}\int_0^1\frac{dx}{\sqrt{1-x^2[1-c\,\ln x]^2}} \quad \text{if} \quad \lambda = 2 \tag{B6}$$

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